

ON THE GENERALIZED ZALCMAN FUNCTIONAL $\lambda a_n^2 - a_{2n-1}$ IN THE CLOSE-TO-CONVEX FAMILY

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ABSTRACT. Let \mathcal{S} denote the class of all functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the unit disk \mathbb{D} . For $f \in \mathcal{S}$, Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for $n \geq 3$. This conjecture has been verified only certain values of n for $f \in \mathcal{S}$ and for all $n \geq 4$ for the class \mathcal{C} of close-to-convex functions (and also for a couple of other classes). In this paper we provide bounds of the generalized Zalcman coefficient functional $|\lambda a_n^2 - a_{2n-1}|$ for functions in \mathcal{C} and for all $n \geq 3$, where λ is a positive constant. In particular, our special case settles the open problem on the Zalcman inequality for $f \in \mathcal{C}$ (i.e. for the case $\lambda = 1$ and $n = 3$).

1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in \mathbb{C} and $\partial\mathbb{D} = \{z : |z| = 1\}$. Let \mathcal{A} denote the class of all analytic functions in \mathbb{D} and $\mathcal{S} \subset \mathcal{A}$ denote the family of all normalized univalent functions f of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function $f \in \mathcal{S}$ is called starlike if $f(\mathbb{D})$ is starlike with respect to the origin. Let \mathcal{S}^* denote the class of all starlike functions in \mathcal{S} . An analytic function f of the form (1) is called close-to-convex if there exists a real number θ and a function $g \in \mathcal{S}^*$ such that $\operatorname{Re} (e^{i\theta} z f'(z)/g(z)) > 0$ for $z \in \mathbb{D}$. Functions in the class \mathcal{C} of all close-to-convex functions are known to be univalent in \mathbb{D} . Geometrically, $f \in \mathcal{C}$ means that the complement of the image-domain $f(\mathbb{D})$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays).

The role of the family \mathcal{S} together with their subfamilies and their importance geometric function theory are well documented, see for example [5, 6, 7, 21, 22]. Several conjectures which imply the Bieberbach conjecture that $|a_n| \leq n$ for each $f \in \mathcal{S}$ have been verified by the de Branges theorem. At that time, as an approach to prove the Bieberbach conjecture (see also [3]), Lawrence Zalcman in 1960 conjectured that the coefficients of \mathcal{S} satisfy the sharp inequality

$$(2) \quad |a_n^2 - a_{2n-1}| \leq (n-1)^2,$$

for each $n \geq 2$ with equality only for the Koebe function $k(z) = z/(1-z)^2$ and its rotation. The conjecture remains open and partial progress was obtained in the sense that it has been verified either for certain subclasses of \mathcal{S} or for certain values of n for the full class \mathcal{S} (eg. see [12] for the case $n = 3$, and [13] for the cases $n = 4, 5, 6$). Analog of this conjecture for several other subclasses of \mathcal{S} has not been attempted except for some cases [1, 2, 14, 15].

Recall that the case $n = 2$ of Zalcman conjecture is the well-known Fekete-Szegő inequality, namely, $|a_2^2 - a_3| \leq 1$ (see [21, Theorem 1.5] and [5, Theorem 3.8]). Moreover, the problem of maximizing $|\lambda a_2^2 - a_3|$ with $\lambda > 0$ for \mathcal{S} ([5, Theorem 3.8]) and for many other subclasses has been

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solved by a number of authors—often for restricted values of λ , see for instance [2, 4, 8, 9, 10, 11]. In [20], Pfluger extended the later result for certain complex values of λ .

In [19], Pfluger pointed out that if $f \in \mathcal{S}$ then the coefficients of $\sqrt{f(z^2)}$ and $1/f(1/z)$ are polynomials in a_j which contain expressions of the form $\lambda a_n^2 - a_{2n-1}$ which is referred to as the generalized Zalcman functional. Thus it is natural to consider the problem of maximizing $|\lambda a_n^2 - a_{2n-1}|$, as a function of the real parameter λ . There are only few results of this type in the literature and that too deal with only $\lambda = 1$ for a few classes of functions f . For example, in [3], Brown and Tsao proved the Zalcman conjecture for $n \geq 3$ for typically real functions (hence for functions with real coefficients in \mathcal{S}) and also for the class \mathcal{S}^* . In fact they have proved it in a general form involving generalized Zalcman functional, and another generalization of the result of Brown and Tsao appeared in [17]. In 1988, Ma [16] provided further evidence in support of the conjecture by verifying the Zalcman inequality (2) (again for the case $\lambda = 1$ only) for the class \mathcal{C} when $n \geq 4$. However, the conjecture remains open in \mathcal{C} for $n = 3$ (see [16, Remarks]).

In this article, we consider the generalized Zalcman conjecture for the class \mathcal{C} and $n \geq 3$. In particular, we solve the conjecture for the family \mathcal{C} when $n = 3$. Also, one of our main results contains the proof of the main result of Ma [16].

We end the section by indicating the recent result on generalized Zalcman conjecture. Let $\mathcal{F}(\alpha)$ denote the family of convex functions of order α with $-1/2 \leq \alpha < 1$, consisting of functions $f \in \mathcal{S}$ satisfying the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{for } z \in \mathbb{D}.$$

We remark that $\mathcal{F}(0) := \mathcal{K}$, the class of all normalized convex univalent functions, i.e. each $f \in \mathcal{K}$ maps \mathbb{D} univalently onto a convex domain. It is known that $\mathcal{F}(\alpha) \subset \mathcal{C}$ for $-1/2 \leq \alpha < 0$, and $\mathcal{F}(\alpha) \subset \mathcal{K}$ for $0 \leq \alpha < 1$. The sharp bound for the quantity $|a_n^2 - a_{2n-1}|$ for the class of convex functions of order $-1/2$ was first discussed in [1] and in recent articles, Li and Ponnusamy [14] (see also Li et al. [15]) obtained sharp bound for the generalized Zalcman functional $|\lambda a_n^2 - a_{2n-1}|$, for the family $\mathcal{F}(\alpha)$.

We now state our main results. It is worth pointing out that all the results below continue to hold if “ $f \in \mathcal{C}$ ” is replaced by “ $f \in \mathcal{S}^*$ ”. This observation shows that our results include a proof of the generalized Zalcman conjecture for the class \mathcal{S}^* and $n \geq 3$.

Theorem 1. *Suppose that $f \in \mathcal{C}$ as in the form (1) and $n \geq 3$.*

(1) *If $\lambda \geq \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$, then*

$$|\lambda a_n^2 - a_{2n-1}| \leq \lambda n^2 - (2n - 1),$$

where the equality holds if $f(z)$ is the Koebe function $z/(1-z)^2$.

(2) *If $\frac{2n}{n^2-n+1} < \lambda < \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$, then*

$$|\lambda a_n^2 - a_{2n-1}| \leq \frac{\lambda [4n(n+1) - (3n^2+1)\lambda] + 4n(2-\lambda)\sqrt{n\lambda(2-\lambda)}}{\lambda [8n - \lambda(n+1)^2]}.$$

(3) *If $0 < \lambda \leq \frac{2n}{n^2-n+1}$, then we have*

$$|\lambda a_n^2 - a_{2n-1}| \leq 2n - 1.$$

Substituting $\lambda = 1$ in Theorem 1(1), it follows easily that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ for all $n \geq 4$ and $f \in \mathcal{C}$. It is worth to state the cases $n = 3, 4$ explicitly.

Corollary 1. *Suppose that $f \in \mathcal{C}$ as in the form (1).*

(1) If $\lambda \geq 1$, then

$$|\lambda a_4^2 - a_7| \leq 16\lambda - 7,$$

where the equality holds if $f(z)$ is the Koebe function $z/(1-z)^2$.

(2) If $\frac{13}{8} < \lambda < 1$, then

$$|\lambda a_4^2 - a_7| \leq \frac{\lambda(80 - 49\lambda) + 32(2 - \lambda)\sqrt{\lambda(2 - \lambda)}}{\lambda(32 - 25\lambda)}.$$

(3) If $0 < \lambda \leq \frac{8}{13}$, then we have $|\lambda a_4^2 - a_7| \leq 7$.

Corollary 2. Suppose that $f \in \mathcal{C}$ as in the form (1).

(1) If $\lambda \geq \frac{9+\sqrt{33}}{12}$, then

$$|\lambda a_3^2 - a_5| \leq 9\lambda - 5,$$

where the equality holds if $f(z)$ is the Koebe function $z/(1-z)^2$.

(2) If $\frac{6}{7} < \lambda < \frac{9+\sqrt{33}}{12}$, then

$$|\lambda a_3^2 - a_5| \leq \frac{\lambda(12 - 7\lambda) + 3(2 - \lambda)\sqrt{3\lambda(2 - \lambda)}}{\lambda(6 - 4\lambda)}.$$

(3) If $0 < \lambda \leq \frac{6}{7}$, then we have $|\lambda a_3^2 - a_5| \leq 5$.

Clearly, Corollary 2(2) gives that if $f \in \mathcal{C}$ is given by (1), then

$$|a_3^2 - a_5| \leq \frac{5 + 3\sqrt{3}}{2} \approx 5.098.$$

Proof of Theorem 1 rely on a number of lemmas. In Section 2, we present three important lemmas which play vital role in the formulation of several lemmas in Section 3. In Section 3, we state and prove several lemmas based on different interval range values of λ . The proof of Theorem 1 will be given in Section 4.

2. PRELIMINARIES AND SOME BASIC LEMMAS

Suppose that X is a linear topological space and that $Y \subset X$. The subset Y is called convex if $tx + (1-t)y \in Y$ whenever $x, y \in Y$ and $0 \leq t \leq 1$. The closed convex hull of Y is defined as the intersection of all closed convex sets containing Y . A point $u \in Y$ is called an extremal point of Y if, for $0 < t < 1$ and $x, y \in Y$, $u = tx + (1-t)y$ implies that $x = y$. The set $\mathcal{E}Y$ consists of all the extremal points of Y (see [7, 18] for a general reference and for many important results on this topic).

Lemma A. [7] Let \mathcal{HC} and \mathcal{EHC} denote the closed convex hull of \mathcal{C} and the set of the extremal points of \mathcal{HC} , respectively. Then \mathcal{HC} consists of all analytic functions represented by

$$f(z) = \int_S \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} d\mu(x, y),$$

where μ is a probability measure on $S = \partial\mathbb{D} \times \partial\mathbb{D}$. The set \mathcal{EHC} consists of the functions given by

$$(3) \quad f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2} = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}y^{n-1} - \frac{n-1}{2}xy^{n-2} \right) z^n,$$

where $|x| = |y| = 1$ and $x \neq y$.

If the family $\mathcal{F} \subset \mathcal{A}$ is convex and $L : \mathcal{A} \rightarrow \mathbb{R}$ is a real-valued functional on \mathcal{A} , then we say that L is convex on \mathcal{F} provided that

$$L(tg_1 + (1-t)g_2) \leq tL(g_1) + (1-t)L(g_2)$$

whenever $g_1, g_2 \in \mathcal{F}$ and $0 \leq t \leq 1$. Since \mathcal{HC} is convex, we have a real-valued, continuous and convex functional on \mathcal{HC} .

Lemma 1. For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is analytic in \mathbb{D} , consider

$$J(g) = \lambda (\operatorname{Re} b_n)^2 - \operatorname{Re} b_{2n-1},$$

where $\lambda > 0$. Then J is a real-valued, continuous and convex functional on \mathcal{HC} .

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be analytic in \mathbb{D} , $0 \leq t \leq 1$ and $G = tg + (1-t)h$. Then, by the definition of J , we have

$$J(G) = \lambda [\operatorname{Re}(tb_n + (1-t)c_n)]^2 - \operatorname{Re}[tb_{2n-1} + (1-t)c_{2n-1}]$$

which may be easily rearranged as

$$J(G) = tJ(g) + (1-t)J(h) - \lambda t(1-t)[\operatorname{Re} b_n - \operatorname{Re} c_n]^2.$$

This gives $J(tg + (1-t)h) \leq tJ(g) + (1-t)J(h)$ and the desired conclusions follow. \square

Since \mathcal{C} is compact, for the functional J defined as in Lemma 1, Theorem 4.6 in [7] yields the following.

Lemma B. $\max\{J(f) : f \in \mathcal{HC}\} = \max\{J(f) : f \in \mathcal{C}\} = \max\{J(f) : f \in \mathcal{EHC}\}.$

By using Lemmas A, 1 and B, we derive the following lemma.

Lemma 2. We have $4\max\{J(f) : f \in \mathcal{C}\} - 4n \leq F_{n,\lambda}(u, v)$, where $F_{n,\lambda}(u, v) =: F(u, v)$ is given by

$$F(u, v) = [(n+1)^2\lambda - 8n]u^2 - 2(n-1)[(n+1)\lambda - 2]uv + 4(n-1)\sqrt{1-u^2}\sqrt{1-v^2} + (n-1)^2\lambda v^2, \\ \text{and } (u, v) \in R =: [-1, 1] \times [-1, 1].$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{EHC}$. Then Lemma A and (3) (with $x = e^{is}$ and $y = e^{it}$) give that

$$a_n = \frac{n+1}{2}e^{i(n-1)t} - \frac{n-1}{2}e^{i[s+(n-2)t]},$$

where $t, s \in [0, 2\pi)$ and $t \neq s$. This representation quickly yields

$$J(f) = \lambda \left\{ \frac{n+1}{2} \cos(n-1)t - \frac{n-1}{2} \cos(s + (n-2)t) \right\}^2 \\ - n \cos 2(n-1)t + (n-1) \cos(s + (2n-3)t).$$

By the identity $\cos 2\theta = 2\cos^2 \theta - 1$ and the addition formula for $\cos(s + (n-2)t + (n-1)t)$, we may rewrite the last expression as

$$J(f) = \left(\frac{(n+1)^2}{4} \lambda - 2n \right) \cos^2(n-1)t \\ - (n-1) \left(\frac{n+1}{2} \lambda - 1 \right) \cos(n-1)t \cos(s + (n-2)t) \\ - (n-1) \sin(n-1)t \sin(s + (n-2)t) + \frac{(n-1)^2}{4} \lambda \cos^2(s + (n-2)t) + n.$$

If we set $\cos(n-1)t = u$ and $\cos(s + (n-2)t) = v$ and use the identity $\sin \theta = \pm \sqrt{1 - \cos^2 \theta}$, then the above equation reduces to

$$4J(f) - 4n \leq F(u, v),$$

where $u, v \in [-1, 1]$. Lemmas A, 1 and B show that the proof is completed. \square

3. MAIN LEMMAS

Throughout $F(u, v) := F_{n,\lambda}(u, v)$ is given by Lemma 2 whereas $G_{n,\lambda}(u, v) =: G(u, v)$ is defined by (5) below, and $R = [-1, 1] \times [-1, 1]$. Observe that $F(u, v) = F(-u, -v)$ and $G(u, v) = G(-u, -v)$.

Lemma 3. *Suppose that $n \geq 3$ and $\lambda \geq \frac{10n-2}{(n+1)^2}$. Then*

$$F(u, v) \leq 4\lambda n^2 - 12n + 4 \quad \text{for } (u, v) \in R,$$

where the equality holds if and only if $(u, v) = (1, -1)$ or $(u, v) = (-1, 1)$.

Proof. The elementary inequality $2ab \leq a^2 + b^2$ gives

$$(4) \quad F(u, v) \leq G(u, v) + 4(n-1),$$

where $G_{n,\lambda}(u, v) =: G(u, v)$ is given by

$$(5) \quad G(u, v) = [\lambda(n+1)^2 - 10n + 2]u^2 - 2(n-1)[\lambda(n+1) - 2]uv + (n-1)[\lambda(n-1) - 2]v^2.$$

We now introduce

$$A(n) = \frac{10n-2}{(n+1)^2} \quad \text{and} \quad B(n) = \frac{2}{n}.$$

Then for $n \geq 3$, we observe that $A(n) > B(n-1) > B(n+1)$, and the functions $A(n)$ and $B(n)$ are monotonically decreasing with respect to n . Using $A(n)$ and $B(n)$, we may write (5) in a convenient form as

$$(6) \quad G(u, v) = (n+1)^2 [\lambda - A(n)] u^2 - 2(n^2 - 1) [\lambda - B(n+1)] uv + (n-1)^2 [\lambda - B(n-1)] v^2.$$

If $n \geq 3$ and $\lambda \geq \frac{10n-2}{(n+1)^2}$, then $\lambda \geq A(n) > B(n-1) > B(n+1)$ and thus,

$$G(u, v) \leq 4\lambda n^2 - 16n + 8,$$

where the equality holds if and only if $(u, v) = (1, -1)$ or $(u, v) = (-1, 1)$. The proof is completed. \square

Lemma 4. *Suppose that $n \geq 3$ and $\frac{6n-2}{n^2+n} \leq \lambda < \frac{10n-2}{(n+1)^2}$. Then*

$$F(u, v) \leq 4\lambda n^2 - 12n + 4 \quad \text{for } (u, v) \in R,$$

where the equality holds if and only if $(u, v) = (1, -1)$ or $(u, v) = (-1, 1)$.

Proof. In view of the inequality (4), it suffices to prove that

$$G(u, v) \leq G(1, -1) = G(-1, 1) = 4\lambda n^2 - 16n + 8.$$

To complete the proof, by assumption, we begin to observe that

$$(7) \quad \frac{2}{n+1} < \frac{2n}{n^2 - n + 1} < \frac{4n-2}{n^2} < \frac{6n-2}{n^2+n} \leq \lambda < \frac{10n-2}{(n+1)^2}.$$

Next, we determine the critical points of $G(u, v)$ and for that we need to consider the equations

$$(8) \quad \begin{cases} \frac{1}{2} \cdot \frac{\partial G(u, v)}{\partial u} = [\lambda(n+1)^2 - 10n + 2] u - (n-1)[\lambda(n+1) - 2] v = 0 \\ \frac{1}{2(n-1)} \cdot \frac{\partial G(u, v)}{\partial v} = -[\lambda(n+1) - 2] u + [\lambda(n-1) - 2] v = 0. \end{cases}$$

The equations have only solution $(u, v) = (0, 0)$, and $G(0, 0) = 0$. We now divide the proof into four cases.

Case 1: $u = 1$.

Since $n \geq 3$ and $\frac{6n-2}{n^2+n} \leq \lambda$, by (7), it is clear that $\lambda(n-1) - 2 > 0$ and thus, the function $\psi(v) = G(1, v)$ on $[-1, 1]$, where

$$(9) \quad \psi(v) = [\lambda(n+1)^2 - 10n + 2] - 2(n-1)[\lambda(n+1) - 2]v + (n-1)[\lambda(n-1) - 2]v^2,$$

attains its maximum at $v = -1$ so that $\psi(v) \leq \psi(-1)$ for $v \in [-1, 1]$.

Case 2: $u = -1$.

By similar reasoning as in Case 1, we can easily show that $G(-1, v) \leq G(-1, 1)$ for $v \in [-1, 1]$.

Case 3: $v = 1$.

In this case, we regard $\phi(u) = G(u, 1)$ as a function of u on the interval $[-1, 1]$. We have

$$(10) \quad \phi(u) = [\lambda(n+1)^2 - 10n + 2]u^2 - 2(n-1)[\lambda(n+1) - 2]u + (n-1)[\lambda(n-1) - 2]$$

so that

$$\phi'(u) = 2[\lambda(n+1)^2 - 10n + 2]u - 2(n-1)[\lambda(n+1) - 2] \quad \text{and} \quad \phi''(u) = 2[\lambda(n+1)^2 - 10n + 2].$$

Solving the equation $\phi'(u) = 0$ gives only solution

$$(11) \quad u_0 = \frac{(n-1)[\lambda(n+1) - 2]}{\lambda(n+1)^2 - 10n + 2}.$$

By assumption $u_0 \leq -1$, where the inequality holds if and only if $\lambda = \frac{6n-2}{n^2+n}$.

If $\lambda = \frac{6n-2}{n^2+n}$, then $u_0 = -1$ is the point of maximum of $\phi(u)$ on $[-1, 1]$, since $\phi''(u) < 0$.

If $\lambda > \frac{6n-2}{n^2+n}$, then $u_0 < -1$ and thus,

$$\phi(u) \leq \max\{\phi(1), \phi(-1)\} = \phi(-1) \quad \text{for } u \in [-1, 1].$$

Case 4: $v = -1$.

By similar reasoning as in Case 3, we can easily see that

$$G(u, -1) \leq G(1, -1) \quad \text{for } u \in [-1, 1].$$

Finally, $G(-1, 1) - G(0, 0) = 4\lambda n^2 - 16n + 8 > 0$, which is obvious, since $\lambda \geq \frac{6n-2}{n^2+n}$. The proof is completed. \square

Lemma 5. Suppose that $n \geq 3$ and $0 < \lambda \leq \frac{2n}{n^2-n+1}$. Then $G(u, v) \leq 0$ for $(u, v) \in R$, where $G(u, v)$ is defined by (5).

Proof. Clearly, we have the chain of inequalities

$$(12) \quad \frac{2}{n+1} < \frac{2}{n} < \frac{2n}{n^2-n+1} < \frac{2}{n-1} < \frac{4n-2}{n^2} < \frac{5n-1}{n^2+n} < \frac{6n-2}{n^2+n} < \frac{8n}{(n+1)^2} < \frac{10n-2}{(n+1)^2}.$$

Case 1: $\lambda = \frac{2}{n+1}$.

Substituting this value of λ in (5), we find that

$$G(u, v) = -4(2n-1)u^2 - \frac{4(n-1)}{n+1}v^2 \leq 0 = G(0, 0).$$

Case 2: $\lambda = \frac{2n}{n^2-n+1}$.

Again, substituting this value of λ in (5), we see that

$$G(u, v) = -\frac{2(n-1)}{n^2-n+1} [(2n-1)u + v]^2 \leq 0 = G(0, 0).$$

Case 3: $\lambda \notin \{\frac{2}{n+1}, \frac{2n}{n^2-n+1}\}$.

From the partial derivatives of $G(u, v)$ determined from (8), we obtain that the equations in (8) have only solution $(u, v) = (0, 0)$, and $G(0, 0) = 0$. As before, we need to divide the proof into four subcases.

Subcase 3(a): $u = 1$.

For this subcase, we consider the function $\psi(v) = G(1, v)$ defined by (9) and obtain that

$$\psi'(v) = -2(n-1) [\lambda(n+1) - 2] + 2(n-1) [\lambda(n-1) - 2] v.$$

We see that $\psi'(v) = 0$ yields the solution v_0 , where

$$v_0 = \frac{\lambda(n+1) - 2}{\lambda(n-1) - 2}, \quad \psi(v_0) = \frac{8[2n - \lambda(n^2 - n + 1)]}{\lambda(n-1) - 2} \quad \text{and} \quad \psi(-1) = 4\lambda n^2 - 16n + 8.$$

A computation implies that $|v_0| \leq 1$ if and only if $0 < \lambda \leq \frac{2}{n}$. Therefore, we have

$$G(1, v) \leq \begin{cases} \frac{8[2n - \lambda(n^2 - n + 1)]}{\lambda(n-1) - 2} & \text{for } 0 < \lambda \leq \frac{2}{n} \\ 4\lambda n^2 - 16n + 8 & \text{for } \frac{2n}{n^2-n+1} > \lambda > \frac{2}{n}. \end{cases}$$

Subcase 3(b): $u = -1$.

By similar reasoning as in Subcase 3(a), we can easily see that the last inequality continues to hold with $G(-1, v)$ instead of $G(1, v)$.

Subcase 3(c): $v = 1$.

For this subcase, we recall the function $\phi(u) = G(u, 1)$ defined by (10). We observe that u_0 defined by (11) has the property (by assumption) that $|u_0| < 1$ for $0 < \lambda < \frac{6n-2}{n^2+n}$ and so, u_0 is the point of maximum for $\phi(u)$ on $[-1, 1]$, which yields that

$$(13) \quad \phi(u) \leq \phi(u_0) = \frac{8(n-1) [\lambda(n^2 - n + 1) - 2n]}{10n - 2 - \lambda(n+1)^2}.$$

Subcase 3(d): $v = -1$.

By similar reasoning as in Subcase 3(c), we can easily prove that the last inequality (13) continues to hold in this case too. Since $0 < \lambda < \frac{2n}{n^2-n+1}$ and $n \geq 3$, we deduce that

$$\frac{8[2n - \lambda(n^2 - n + 1)]}{\lambda(n-1) - 2} \leq 0, \quad 4\lambda n^2 - 16n + 8 < 0, \quad \text{and} \quad \frac{8(n-1) [\lambda(n^2 - n + 1) - 2n]}{10n - 2 - \lambda(n+1)^2} \leq 0.$$

Finally, the above facts imply the desired conclusion of the lemma. \square

Lemma 6. Suppose that $n \geq 3$, $\frac{2n}{n^2-n+1} < \lambda < \frac{6n-2}{n^2+n}$ and $F(u, v) := F_{n,\lambda}(u, v)$ is given by Lemma 2. Then

$$\max \{F(u, v) : (u, v) \in \partial R\} \leq A_{n,\lambda},$$

where

$$(14) \quad A_{n,\lambda} = \begin{cases} \frac{4(n-1)^2 [\lambda(n-1) + 1]}{8n - (n+1)^2 \lambda} & \text{for } \frac{2n}{n^2-n+1} < \lambda \leq \frac{5n-1}{n^2+n} \\ 4\lambda n^2 - 12n + 4 & \text{for } \frac{5n-1}{n^2+n} < \lambda < \frac{6n-2}{n^2+n}. \end{cases}$$

Proof. We continue to use the chain of inequalities given by (12) and as before, the proof is divided into four cases.

Case 1: $u = 1$.

For this case, $\lambda > \frac{2n}{n^2-n+1} > \frac{2}{n+1}$, the function $\Psi(v) := F(1, v)$ defined by

$$\Psi(v) = [(n+1)^2 \lambda - 8n] - 2(n-1)[(n+1)\lambda - 2]v + (n-1)^2 \lambda v^2$$

attains its maximum at $v = -1$. This observation yields that

$$\Psi(v) \leq \Psi(-1) = 4\lambda n^2 - 12n + 4.$$

Case 2: $u = -1$.

By similar reasoning as in Case 1, we conclude that $F(-1, v) \leq 4\lambda n^2 - 12n + 4$.

Case 3: $v = 1$.

In this case, we need to consider the function $\Phi(u) := F(u, 1)$ defined by

$$\Phi(u) = [(n+1)^2 \lambda - 8n]u^2 - 2(n-1)[(n+1)\lambda - 2]u + (n-1)^2 \lambda.$$

Clearly, the only solution u_0 to the equation $\Phi'(u) = 0$ is given by

$$u_0 = \frac{(n-1)[\lambda(n+1) - 2]}{\lambda(n+1)^2 - 8n}.$$

Moreover, $|u_0| \leq 1$ if and only if

$$[(n^2 + n)\lambda - (5n - 1)][(n+1)\lambda - (3n+1)] \geq 0.$$

Also, we have

$$\Phi(u_0) = F(u_0, 1) = \frac{4(n-1)^2 [\lambda(n-1) + 1]}{8n - (n+1)^2 \lambda}$$

and

$$\Phi(-1) = F(-1, 1) = F(1, -1) = 4\lambda n^2 - 12n + 4.$$

Since $\lambda < \frac{6n-2}{n^2+n} < \frac{3n+1}{n+1}$ and $\frac{5n-1}{n^2+n} < \frac{6n-2}{n^2+n}$, the above facts show that

$$|u_0| \begin{cases} \leq 1 & \text{for } \frac{2n}{n^2-n+1} < \lambda \leq \frac{5n-1}{n^2+n} \\ > 1 & \text{for } \frac{5n-1}{n^2+n} < \lambda < \frac{6n-2}{n^2+n}. \end{cases}$$

It follows that $F(u, 1) \leq A_{n,\lambda}$, where $A_{n,\lambda}$ is given by (14).

Case 4. $v = -1$.

Again, by similar reasoning as in Case 3, we can prove that $F(u, -1) \leq A_{n,\lambda}$. Moreover, by a computation, for $\lambda < \frac{6n-2}{n^2+n}$, we have

$$\Phi(-1) - \Phi(u_0) = 4\lambda n^2 - 12n + 4 - \frac{4(n-1)^2 [\lambda(n-1) + 1]}{8n - (n+1)^2 \lambda} = -\frac{4[n(n+1)\lambda - (5n-1)]^2}{8n - (n+1)^2 \lambda} \leq 0.$$

The desired conclusion follows if we use the above facts and combine the four cases. \square

Lemma 7. Suppose that $n \geq 3$, $\frac{2n}{n^2-n+1} < \lambda < \frac{6n-2}{n^2+n}$, and $F(u, v) := F_{n,\lambda}(u, v)$ is given by Lemma 2. Then the critical points of $F(u, v)$ are $(0, 0)$ and (u, v) , where (u, v) satisfies

$$(15) \quad v^2 = \frac{[\sqrt{n\lambda} + \sqrt{2-\lambda}]^2[(n-1)\lambda - \sqrt{\lambda n(2-\lambda)}]}{(n-1)^2\lambda^2},$$

$$(16) \quad uv = \frac{[(n+1)\lambda - 2][\lambda(n-1) - \sqrt{\lambda n(2-\lambda)}]}{(n-1)\lambda[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}]},$$

and

$$(17) \quad u^2 = \frac{[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)} + 2][(n-1)\lambda - \sqrt{\lambda n(2-\lambda)}]}{[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}]^2}.$$

Proof. The critical points of $F(u, v)$ are the solutions of the equations

$$(18) \quad \frac{\partial F(u, v)}{\partial u} = 0 \quad \text{and} \quad \frac{\partial F(u, v)}{\partial v} = 0,$$

which is equivalent to solving the pair of equations

$$(19) \quad \begin{cases} [(n+1)^2\lambda - 8n]u - (n-1)[(n+1)\lambda - 2]v = 2(n-1)u\frac{\sqrt{1-v^2}}{\sqrt{1-u^2}}, \\ -[(n+1)\lambda - 2]u + (n-1)\lambda v = 2v\frac{\sqrt{1-u^2}}{\sqrt{1-v^2}}. \end{cases}$$

It is obvious that $(u, v) = (0, 0)$ is a solution of (19). We now assume that $(u, v) \neq (0, 0)$. It follows from (19) that

$$(20) \quad a(n)\frac{u^2}{v^2} + b(n)\frac{u}{v} + c(n) = 0,$$

where

$$\begin{aligned} a(n) &= [(n+1)^2\lambda - 8n][(n+1)\lambda - 2] \\ b(n) &= -2\lambda(n-1)[(n+1)^2\lambda - 2(3n+1)] \quad \text{and} \\ c(n) &= \lambda(n-1)^2[(n+1)\lambda - 2]. \end{aligned}$$

We consider the discriminant Δ of the quadratic equation (20) in the variable u/v . By a computation, we have

$$\Delta = b^2(n) - 4a(n)c(n) = 64(n-1)^2\lambda n(2-\lambda) > 0,$$

since $n \geq 3$ and $\frac{2n}{n^2-n+1} < \lambda < \frac{6n-2}{n^2+n}$. Thus, (20) has two solutions

$$\frac{u}{v} = (n-1) \left[\frac{\lambda[(n+1)^2\lambda - 2(3n+1)] \pm 4\sqrt{\lambda n(2-\lambda)}}{[(n+1)^2\lambda - 8n][(n+1)\lambda - 2]} \right].$$

In the case

$$u = (n-1) \left[\frac{\lambda[(n+1)^2\lambda - 2(3n+1)] - 4\sqrt{\lambda n(2-\lambda)}}{[(n+1)^2\lambda - 8n][(n+1)\lambda - 2]} \right] v,$$

it follows from the second identity of (19) that

$$\begin{aligned} 2\sqrt{\frac{1-u^2}{1-v^2}} &= (n-1) \left[\lambda - \frac{\lambda[(n+1)^2\lambda - 2(3n+1)] - 4\sqrt{\lambda n(2-\lambda)}}{(n+1)^2\lambda - 8n} \right] \\ &= -\frac{2(n-1)[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}]}{(n+1)^2\lambda - 8n}. \end{aligned}$$

The right of the last equation is less than zero, since

$$\lambda[(n+1)^2\lambda - 8n] = [\lambda(n-1) + 2\sqrt{\lambda n(2-\lambda)}] [\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}] < 0.$$

This is clearly a contradiction. In view of this observation, we only need to consider the case

$$u = (n-1) \left[\frac{\lambda[(n+1)^2\lambda - 2(3n+1)] + 4\sqrt{\lambda n(2-\lambda)}}{[(n+1)^2\lambda - 8n][(n+1)\lambda - 2]} \right] v,$$

which may be rewritten as

$$(21) \quad u = (n-1) \left[\frac{\lambda[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)} + 2]}{[(n+1)\lambda - 2][\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}]} \right] v.$$

It follows quickly from (21) and the second identity of (19) that

$$(22) \quad \sqrt{\frac{1-u^2}{1-v^2}} = \frac{(n-1)\lambda}{2\sqrt{\lambda n(2-\lambda)} - \lambda(n-1)}.$$

Substituting (21) in (22) yields

$$1 - \frac{(n-1)^2\lambda^2[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)} + 2]^2}{[(n+1)\lambda - 2]^2[\lambda(n-1) - 2\sqrt{\lambda n(2-\lambda)}]^2} v^2 = \frac{(n-1)^2\lambda^2}{[2\sqrt{\lambda n(2-\lambda)} - \lambda(n-1)]^2} (1 - v^2)$$

which after simplification gives

$$v^2 = \frac{[(n+1)\lambda - 2]^2}{(n-1)^2\lambda^2} \cdot \frac{(n-1)\lambda - \sqrt{\lambda n(2-\lambda)}}{[\sqrt{n\lambda} - \sqrt{2-\lambda}]^2}.$$

This is the same as (15) and observe that the right side expression in the above form of v^2 is clearly positive, because

$$(n-1)^2\lambda^2 - \lambda n(2-\lambda) = \lambda[(n^2 - n + 1)\lambda - 2n] > 0.$$

Using (15), (21) easily gives (16) and (17). The proof of the lemma is complete. \square

Lemma 8. Suppose that $F(u, v) := F_{n,\lambda}(u, v)$ is given by Lemma 2. Then we have the following:

- (1) If $n \geq 3$ and $\frac{3n+\sqrt{5n^2-4n}}{n^2+n} \leq \lambda < \frac{6n-2}{n^2+n}$, then the only critical point of $F(u, v)$ for $(u, v) \in (-1, 1) \times (-1, 1)$ is $(0, 0)$, and that $F(0, 0) = 4(n-1)$.
- (2) If $n \geq 3$ and $\frac{2n}{n^2-n+1} < \lambda < \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$, then there are three critical points for $F(u, v)$ on $(-1, 1) \times (-1, 1)$, namely, $(0, 0)$ and (u_i, v_i) ($i = 1, 2$) which satisfy (15), (16) and (17). Moreover,

$$(23) \quad F(u_1, v_1) = F(u_2, v_2) = \frac{4\lambda(n-1)[\lambda(n^2+1) - 4n] + 16n(2-\lambda)\sqrt{n\lambda(2-\lambda)}}{\lambda[8n - \lambda(n+1)^2]},$$

and $F(u_1, v_1) > F(0, 0)$.

Proof. For $n \geq 3$ and $\frac{2n}{n^2-n+1} < \lambda < \frac{6n-2}{n^2+n}$, Lemma 7 and (22) yield $v^2 > u^2$. Equation (15) shows that $v^2 < 1$ if and only if

$$(24) \quad 2\lambda[n\lambda - (n+1)] + [(n-1)\lambda - 2]\sqrt{\lambda n(2-\lambda)} < 0.$$

By assumption, we have $n\lambda - (n+1) < 0$ which implies that (24) holds if $\frac{2n}{n^2-n+1} < \lambda \leq \frac{2}{n-1}$.

Thus, for $\frac{2}{n-1} < \lambda < \frac{6n-2}{n^2+n}$, (24) holds if and only if

$$A =: 4\lambda[n\lambda - (n+1)]^2 - [(n-1)\lambda - 2]^2 n(2-\lambda) > 0.$$

By simplification, we find that

$$\begin{aligned} A &= n(n+1)^2\lambda^3 - 2n(n+1)(n+3)\lambda^2 + 4(3n^2+n+1)\lambda - 8n \\ &= [n(n+1)\lambda^2 - 6n\lambda + 4][(n+1)\lambda - 2n] \\ &= n(n+1) \left(\lambda - \frac{3n - \sqrt{5n^2 - 4n}}{n^2 + n} \right) \left(\lambda - \frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n} \right) [(n+1)\lambda - 2n]. \end{aligned}$$

Consequently, since $\frac{2}{n-1} < \lambda < \frac{6n-2}{n^2+n}$ and

$$\frac{3n - \sqrt{5n^2 - 4n}}{n^2 + n} < \frac{2}{n-1} < \frac{5n-1}{n^2+n} < \frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n} < \frac{6n-2}{n^2+n} < \frac{2n}{n+1},$$

the above facts imply that $A > 0$ if and only if

$$\frac{2}{n-1} < \lambda < \frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n}.$$

We thus have shown that $1 > v^2 > u^2 > 0$ if

$$\frac{2n}{n^2-n+1} < \lambda < \frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n},$$

whereas $v^2 \geq 1$ if

$$\frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n} \leq \lambda < \frac{6n-2}{n^2+n}.$$

Therefore, if

$$\frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n} \leq \lambda < \frac{6n-2}{n^2+n},$$

there exists exactly one solution of the equation (18) for $(u, v) \in (-1, 1) \times (-1, 1)$ and this solution is $(0, 0)$; if

$$\frac{2n}{n^2-n+1} < \lambda < \frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n},$$

there exist three solutions of the equation (18) for $(u, v) \in (-1, 1) \times (-1, 1)$, where one of them is $(0, 0)$ while the other two are (u_i, v_i) ($i = 1, 2$) which satisfy (15), (16) and (17).

By a routine computation, the rest of the proof follows and we complete the proof. \square

Lemma 9. Let $n \geq 3$ and $R = [-1, 1] \times [-1, 1]$. Then we have the following:

- (1) If $\frac{3n + \sqrt{5n^2 - 4n}}{n^2 + n} \leq \lambda < \frac{6n-2}{n^2+n}$, then $F(u, v) \leq 4\lambda n^2 - 12n + 4$ for $(u, v) \in R$.
- (2) If $\frac{2n}{n^2-n+1} < \lambda < \frac{3n + \sqrt{5n^2 - 4n}}{n^2+n}$, then $F(u, v) \leq F(u_1, v_1)$.

Here $F(u, v) := F_{n,\lambda}(u, v)$ is given by Lemma 2 and $F(u_1, v_1)$ is given by (23).

Proof. If $\frac{3n+\sqrt{5n^2-4n}}{n^2+n} \leq \lambda < \frac{6n-2}{n^2+n}$, Lemmas 6 and 8 show that for $(u, v) \in R$, we have

$$F(u, v) \leq \max \{4\lambda n^2 - 12n + 4, 4(n-1)\} = 4\lambda n^2 - 12n + 4.$$

If $\frac{2n}{n^2-n+1} < \lambda < \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$, then we divide the proof into the following two cases.

Case 1: $\frac{5n-1}{n^2+n} < \lambda < \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$.

It follows from Lemmas 6 and 8 that

$$F(u, v) \leq \max \{4\lambda n^2 - 12n + 4, F(u_1, v_1)\} = F(u_1, v_1),$$

for

$$\begin{aligned} & \lambda \frac{8n - (n+1)^2 \lambda}{4n} [F(u_1, v_1) - (4\lambda n^2 - 12n + 4)] \\ &= \lambda [n(n+1)^2 \lambda^2 - 2n(5n+3)\lambda + 4(5n-1)] + 4(2-\lambda)\sqrt{n\lambda(2-\lambda)} \\ &> \frac{\sqrt{n\lambda(2-\lambda)}}{n-1} [n(n+1)^2 \lambda^2 - 2(5n^2+5n-2)\lambda + (28n-12)] \\ &\geq \frac{\sqrt{n\lambda(2-\lambda)}}{(n-1)n(n+1)^2} (3n^4 - 6n^3 - n^2 + 8n - 4) > 0 \end{aligned}$$

since $n \geq 3$, $\frac{5n-1}{n^2+n} < \lambda < \frac{3n+\sqrt{5n^2-4n}}{n^2+n}$ and $\lambda(n-1) > \sqrt{n\lambda(2-\lambda)}$.

Case 2: $\frac{2n}{n^2-n+1} < \lambda \leq \frac{5n-1}{n^2+n}$.

Lemmas 6 and 8 imply that

$$F(u, v) \leq \max \left\{ \frac{4(n-1)^2 [\lambda(n-1) + 1]}{8n - (n+1)^2 \lambda}, F(u_1, v_1) \right\} = F(u_1, v_1),$$

for

$$\begin{aligned} & \lambda \frac{8n - (n+1)^2 \lambda}{4} \left[F(u_1, v_1) - \frac{4(n-1)^2 [\lambda(n-1) + 1]}{8n - (n+1)^2 \lambda} \right] \\ &= \lambda(n-1)(2n\lambda - 5n + 1) + 4n(2-\lambda)\sqrt{n\lambda(2-\lambda)} \\ &> \sqrt{n\lambda(2-\lambda)}(2n\lambda - 5n + 1 + 8n - 4n\lambda) \\ &= 2n\sqrt{n\lambda(2-\lambda)} \left(\frac{3n+1}{2n} - \lambda \right) > 0, \end{aligned}$$

since $n \geq 3$, $\frac{2n}{n^2-n+1} < \lambda \leq \frac{5n-1}{n^2+n}$ and $\lambda(n-1) > \sqrt{n\lambda(2-\lambda)}$. The proof is completed. \square

4. PROOF OF THEOREM 1

Suppose that $f \in \mathcal{C}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Since $|\lambda a_n^2 - a_{2n-1}|$ is invariant under rotations, we consider instead the problem of maximizing the functional $\operatorname{Re}(\lambda a_n^2 - a_{2n-1})$. Moreover, we find that

$$\operatorname{Re}(\lambda a_n^2 - a_{2n-1}) = \lambda (\operatorname{Re} a_n)^2 - \lambda (\operatorname{Im} a_n)^2 - \operatorname{Re} a_{2n-1} \leq J(f) = \lambda (\operatorname{Re} a_n)^2 - \operatorname{Re} a_{2n-1}.$$

It follows from Lemmas 2, 3, 4 and 9(1) that for $n \geq 3$,

$$4J(f) \leq 4n + F(u, v) \leq 4n + (4\lambda n^2 - 12n + 4) = 4[\lambda n^2 - (2n-1)]$$

and the desired conclusion of Theorem 1(1) follows.

With the same reasoning as above, Lemmas 2 and 9(2) give Theorem 1(2).

Finally, by the inequality (4), Lemmas 2 and 5, we obtain that, for $0 < \lambda \leq \frac{2n}{n^2-n+1}$,

$$4J(f) \leq 4n + 4(n-1) + G(u, v) \leq 4(2n-1)$$

and Theorem 1(3) follows. \square

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